Journal of Pure and Applied Algebra 108 (1996) 279–299

JOURNAL OF PURE AND APPLIED ALGEBRA

Relative liftings

Dorin Popescu

Faculty of Mathematics, Bucharest University, Bucharest 70109, Romania

Communicated by C.A. Weibel; received 9 March 1993; revised 18 May 1994

Abstract

Let (Λ, m) be a complete Cohen–Macaulay local ring x a system of parameters of Λ and M a finite $\bar{\Lambda} := \Lambda/(x)$-module. If $\mathrm{Ext}^2_{\bar{\Lambda}}(M, M) = 0$ then there exists a maximal Cohen–Macaulay Λ-module L such that $L/xL \cong M$ by a result of Auslander–Ding–Solberg. Here we investigate the problem of finding a generalized Cohen–Macaulay Λ-module T such that $T/xT \cong M$. If Λ is regular and $(x) \neq m$ then we hope that our procedure can be useful for some bundle constructions.

0. Introduction

Let (Λ, m, k) be a Noetherian local ring and $x = (x_1, \ldots, x_r)$ a regular system of elements of Λ. A lifting of a finite $\bar{\Lambda}$-module M is a Λ-module L such that

(1) $L/xL \cong M$,

(2) x is a L-sequence.

If Λ is complete and $\mathrm{Ext}^2_{\bar{\Lambda}}(M, M) = 0$ then M is liftable to Λ, i.e. there exists a lifting L of M to Λ (see [1, (1.6)]).

If Λ is a Cohen–Macaulay local ring and x is a system of parameters of Λ then x is a Λ-sequence and M is liftable to Λ if and only if there exists a maximal Cohen–Macaulay Λ-module L such that $L/xL \cong M$. Thus the finite $\bar{\Lambda}$-modules liftable to Λ are exactly the modules from the image of the base change functor

$$F : \mathrm{MCM}(\Lambda) \to \mathrm{Mod}\, \bar{\Lambda}$$

defined on the category of maximal Cohen–Macaulay modules by $L \to L/xL$. So the quoted result from [1] gives an idea about how big is the image of F. If Λ is an excellent Henselian isolated singularity containing a field and k is perfect, or $[k : k^p] < \infty$ when $p := \mathrm{char}\, k > 0$, then there exists an integer $t \gg 0$ such that F is an embedding providing x is chosen in m^t (see [10, Ch. 6; 6, (4.8); 8, (2.8); 7]). Thus we may reduce the description of MCM (Λ) to the description of Im F, where the result from [1] could be helpful.

SSDI 0022-4049(94)00106-5

If Λ is regular then all maximal Cohen–Macaulay Λ-modules are free and usually we are looking to a bigger category of Λ-modules – the bundles i.e. the category of finite Λ-modules, which are free on the punctured spectrum of Λ. More generally if Λ is a Cohen–Macaulay ring and s is a positive integer, let $\mathscr{C}_s(\Lambda)$ be the category of finite Λ-modules E for which $m^s\, H^i_m(E) = 0$, $i \neq \dim \Lambda = \dim E$. By [2, (3.15)] the base change functor

$$G: \mathscr{C}_s(\Lambda) \to \operatorname{Mod} \bar{\Lambda}$$

defines an embedding in the same conditions as F above. Again it will be nice to give an idea of how big is $\operatorname{Im} G$ and so to study finite $\bar{\Lambda}$-modules M which are "liftable" to $\mathscr{C}_s(\Lambda)$, i.e. for which there exists a Λ-module E such that

(1′) $E/xE \cong M$,

(2′) x is a m^s-weak E-sequence, providing $(x) \subset m^{2s}$ (see [9, Appendix 10; 13]).

In this paper we give sufficient conditions for a $\bar{\Lambda}$-module M to be "liftable" to $\mathscr{C}_s(\Lambda)$ in terms of vanishing of some Ext-groups (see Corollary 3.4 for $s = 1$; when $s > 1$ apply Theorem 3.3 for the case $I = m^s$ and $(x) \subset I^2$). The proofs follow [1, (1.6)] in the frame of some weaker notions of liftability – the so-called *relative* (resp. *relative*★) liftable modules, i.e. finite $\bar{\Lambda}$-modules M for which there exists a Λ-module E such that (1′) holds and

(2″) x is a relative (resp. relative★) E-sequence, (see Section 1, or [3; 5; 4, Section 5]).

The relative E-sequence seems to have a nice behaviour with respect to the Koszul complex (see Lemma 1.3) and most of the Auslander–Ding–Solberg theory (see [1]) can be extended in this frame (see Proposition 2.3 and Corollary 2.4). However we are able to state good sufficient conditions for liftability only in the more restrictive frame of relative★ liftability (see Theorems 2.9 and 3.5). If $r = 1$ then both notions coincide and then Theorem 3.6 says that if Λ is a complete local ring, $x \in \Lambda$ a regular element and T_1 a finite $\Lambda_1 := \Lambda/(x^2)$-module such that $N := ((0):\, x)_{T_1}/xT_1$, $M := T_1/xT_1$ satisfy

$$\operatorname{Ext}^1_{\bar{\Lambda}}(N, xT_1) = \operatorname{Ext}^2_{\bar{\Lambda}}(M, xT_1) = 0,$$

then T_1 is relative★ liftable to Λ. If T_1 is an infinitesimal lifting of M to Λ_1 then $((0):\, x)_{T_1} = xT_1$, i.e. $N = 0$ and $M \cong xT_1$. Thus the conditions above reduce to $\operatorname{Ext}^2_{\bar{\Lambda}}(M, M) = 0$, which remind us the Auslander–Ding–Solberg result [1, (1.6)].

Now if x is a system of parameters in Λ and also a relative★ E-sequence then x is a (x)-weak E-sequence (see [9] Appendix and the proof of our Theorem 3.3). This does not imply that length $(H^i_m(E)) < \infty$ for all $i \neq \dim \Lambda$ (it is true for $i = 0$ because we may obtain $(x)H^0_m(E) = 0$, but nothing is known when $i > 0$). As we already said above we need to show that x is a I-weak E-sequence for a certain m-primary ideal I such that $(x) \subset I^2$ (see [9, Appendix 12, 13]). For this purpose we are forced to consider a slightly more general notion the so-called the relative (resp. relative★) (x, I)-liftable $\bar{\Lambda}$-module. The first two sections study the infinitesimal relative (resp. relative★)(x, I)-liftings following the ideas from [1]. Our Lemma 3.1 is just a variant of [1, Theorem (1.2)]. Theorem 3.2 gives sufficient conditions for the existence of relative* (x, I)-liftings which are basical for our main result Theorem 3.3. If $I = (x)$ Theorem 3.2 has a stronger variant in Theorem 3.5 but this one has no applications to generalized Cohen–Macaulay modules. In Propositions 2.12 and 3.7 we reobtain some particular results from [1] using our frame. We end our paper with some examples of modules which are relative* liftable but not liftable.

1. Infinitesimal relative liftings

Let (Λ, m) be a Noetherian local ring, $x = (x_1, \dots, x_r)$ a regular system of elements in Λ, $J = (x_1, \dots, x_r)$, $I \supset J$ an ideal of Λ, $\Lambda_s := \Lambda/(x_1, \dots, x_s)$, $1 \le s \le r$, $\Lambda_r = \bar{\Lambda} = \Lambda/(x)$, $\Lambda_i := \Lambda/IJ^i \in \mathbb{N}$ and M a finite $\bar{\Lambda}$-module. A Λ-module L is a *relative* (x, I)*-lifting* of M to Λ if

(1) $L/JL \cong M$,

(2) x is a relative L-sequence with respect to I, i.e. for all s, $0 \le s < r$ it holds

$$((x_1, \dots, x_s)IL : x_{s+1})_L \cap IL = (x_1, \dots, x_s)L.$$

L is called a *relative** (x, I)*-lifting* of M to Λ if (1) holds and

(3) x is a relative* L-sequence with respect to I, i.e. for all s, $0 \le s < r$ it holds

$$((x_1, \dots, x_s)L : x_{s+1})_L \cap IL = (x_1, \dots, x_s)L$$

(when $I = J$ these conditions were introduced by Fiorentini [3] and especially (3) is studied in many papers for e.g. [5; 4, Section 5]). M is *relative* (resp. *relative**) (x, I)*-liftable* to Λ if it has a relative (resp. relative*) (x, I)*-lifting* to Λ. Clearly a relative* (x, I)-lifting to Λ is also a relative (x, I)-lifting to Λ.

A finite Λ_{i+1}-module E is an *infinitesimal relative* (x, I)*-lifting* of a Λ_i-module T to Λ_{i+1} if

(1′) $E/IJ^iE \cong T$,

(2′) $((x_1, \dots, x_s)IE : x_{s+1})_E \cap IE = (x_1, \dots, x_s)E + IJ^iE$, for all s, $0 \le s < r$.

E is called an *infinitesimal relative* $^\star(x, I)$*-lifting* of T to Λ_{i+1} if (1′) holds and

(3′) $((x_1, \dots, x_s)E : x_{s+1})_E \cap IE = (x_1, \dots, x_s)E + IJ^iE$, for all s, $0 \le s < r$.

E is called an *infinitesimal lifting* of M to Λ_{i+1} if E is an infinitesimal relative* (x, Λ)-lifting of M to Λ_{i+1} (this is the usual notion, see [1]).

Remark 1.1. Let L_1 be an arbitrary finite Λ_1-module with $L_1/JL_1 \cong M$. Then L_1 is an infinitesimal relative (x, I)-lifting of M to Λ_1 if $I = (x)$. If $I \neq (x)$ we may not have $L_1/IL_1 \cong M$ but (3′) holds in this case. However we may have finite Λ_2-modules L_2 with $L_2/IJL_2 \cong L_1$ which are not infinitesimal relative liftings of L_1 to Λ_2. For example, if k is a field, $\Lambda = k[[Y]]$, $x = Y$, $I = (x)$ then $L_2 = \Lambda_2[[Z]]/(x^2Z, Z^4)$ is not an infinitesimal relative lifting of $L_1 = L_2/x^2L_2 \cong \Lambda_1[[Z]]/(Z^4)$ because $((0):x)_{L_2} \cap xL_2 = (x^2, xZ)_{L_2} \neq x^2\ L_2$.

We now describe relative and relative★ sequences in terms of Koszul complexes. These results will be needed in the next sections. Let $s \in \mathbb{N}$, $1 \le s \le r$ and

$$0 \to \Lambda \xrightarrow{\delta_s^{(s)}} \Lambda^s \to \cdots \to \Lambda^{\binom{s}{2}} \xrightarrow{\delta_2^{(s)}} \Lambda^s \xrightarrow{\delta_1^{(s)}} \Lambda \to \Lambda_1 \to 0$$

be the Koszul complex defined by $(x_1, \ldots, x_s)$. Let E be a finite Λ_{i+1}-module, $U_s(E) = \operatorname{Ker}(\delta_1^{(s)} \otimes E)$, $V_s(E) = \operatorname{Im}(\delta_2^{(s)} \otimes E)$. Clearly, $U_1(E) = ((0): x_1)_E$, $V_1(E) = 0$, $V_s(E) \subset JE^s$, $(V_{s-1}(E)|0) \subset V_s(E)$, $(U_{s-1}(E)|0) \subset U_s(E)$ and $\operatorname{Tor}_1^{\Lambda}(\bar{\Lambda}, E) \cong U_r(E)/V_r(E)$.

Lemma 1.2. *Let $s \ge 2$ and $u = (u_1| \ldots |u_s)$ be an element from $U_s(E)$ such that $u_s \in (x_1, \ldots, x_{s-1})E + IJ^iE$. Then there exists an element $u' \in U_{s-1}(E)$ such that $u - (u'|0) \in V_s(E) + IJ^iE^s$.*

Proof. *Let $u \in U_s(E)$* be such that $u_s = \sum_{t=1}^{s-1} x_t v_t + w$ *for some $v_t \in E$, $w \in IJ^iE$.* Then

$$u'' := u - (-x_s|0 \ldots 0|x_1)v_1 - \cdots - (0 \ldots 0| -x_s|x_{s-1})v_{s-1} - (0 \ldots 0|w)$$

has the form $(u'|0)$ for an element $u' \in U_{s-1}(E)$ which certainly works. □

Lemma 1.3. *The following statements are equivalent:*

(1) *for every s, $1 \le s \le r$ it holds*

$$((x_1, \ldots, x_{s-1})IE : x_s)_E \cap IE = (x_1, \ldots, x_{s-1})E + IJ^iE,$$

(2) *for every s, $1 \le s \le r$,*

$$U_s(E) \cap IE^s = V_s(E) + IJ^iE^s.$$

Proof. (1) ⇒ (2): Induct on s. If $s = 1$, then (2) says that $((0): x_1)E \cap IE = IJ^iE$ which is exactly (1). Suppose now $s > 1$ and let $u = (u_1| \ldots |u_s)$ be an element from $U_s(E) \cap IE^s$. We have $x_s u_s \in (x_1, \ldots, x_{s-1})IE$ and $u_s \in IE$. By (1) we get $u_s \in (x_1, \ldots, x_{s-1})E + IJ^iE$. Using Lemma 1.2 we find $u' \in U_{s-1}(E)$ such that

$$u - (u'|0) \in V_s(E) + IJ^iE^s \subset IE^s.$$

In particular $u' \in U_{s-1}(E) \cap IE^{s-1}$. By induction hypothesis, it follows $u' \in V_{s-1}(E) + IJ^iE^{s-1}$ and so $u \in V_s(E) + IJ^iE^s$. Thus $\subset$ holds in (2), the other inclusion being trivial.

(2) ⇒ (1): Let s, $1 \le s \le r$ (case $s = 1$ was already done) and $\alpha \in IE$ be such that $x_s\alpha \in (x_1, \ldots, x_{s-1})IE$. Thus we have

$$x_s\alpha = \sum_{t=1}^{s-1} x_t\beta_t$$

for some $\beta_t \in IE$. By (2) the element $\gamma := (\beta_1 \ldots \beta_{s-1}| -\alpha) \in U_s(E) \cap IE^s$ belongs to $V_s(E) + IJ^iE^s$, i.e. the element γ coincides with

$$(-x_s|0 \ldots 0|x_1)\rho_1 + \cdots + (0 \ldots 0| -x_s|x_{s-1})\rho_{s-1} + (-x_{s-1}|0 \ldots 0|x_1|0)\rho_s + \cdots$$

modulo IJ^iE^s for some $\rho_t \in E$. Since only the first $(s-1)$ tuples have the nonzero elements on the last position, we get

$$\alpha + \sum_{t=1}^{s-1} x_t \rho_t \in IJ^iE,$$

i.e. $\alpha \in (x_1, \dots, x_{s-1})E + IJ^iE$. Thus $\subset$ holds in (1), the other inclusion being trivial. □

Proposition 1.4. *Let E be a finite Λ_{i+1}-module and $T = E/IJ^iE$. Then*

(1) E is an infinitesimal relative (x,I)-lifting of T to Λ_{i+1} if and only if for every s, $1 \le s \le r$ it holds

$$U_s(E) \cap IE^s = V_s(E) + IJ^iE^s,$$

(2) E is an infinitesimal relative★ (x,I)-lifting of T to Λ_{i+1} if and only if for every s, $1 \le s \le r$ the A_s-module $E_s := A_s \otimes_\Lambda E$ is an infinitesimal relative $((x_{s+1}, \dots, x_r), IA_s)$-lifting of $T_s := A_s \otimes_\Lambda T$ to $A_s \otimes_\Lambda \Lambda_{i+1}$.

Proof. (1) follows from Lemma 1.3. For (2) it is enough to see that given s, $0 \le s < r$ the following statements are equivalent:

(a) $(x_1, \dots, x_s)E : x_{s+1})_E \cap IE = (x_1, \dots, x_s)E + IJ^iE$,

(b) $((0) : x_{s+1})_{E_s} \cap IE_s = IJ^iE^s$,

the second follows when E_s is an infinitesimal relative $((x_{s+1}, \dots, x_r), IA_s)$-lifting of T_s. □

Lemma 1.5. *Let E be an infinitesimal relative (x,I)-lifting of $T := E/IJ^iE$ to Λ_{i+1} and $f : E \to T$ the canonical surjection. The assignment $f(\alpha) \to (\delta_1^{(r)} \otimes E)(\alpha)$, $\alpha \in E^r$ defines a surjective Λ-morphism $\rho : T^r \to JE$ inducing an isomorphism $\bar{\rho} : T^r/f(U_r(E)) \to JE$. In particular $IJ^jT^r/IJ^jT^r \cap V_r(T) \cong IJ^{j+1}E$ for $1 \le j < i$.*

Proof. ρ is really a map because if $f(\alpha) = f(\beta)$ for some $\alpha, \beta \in E^r$ then $(\delta_1^{(r)} \otimes E)(\alpha - \beta) \subset J(IJ^iE) = 0$. If $(\delta_1^{(r)} \otimes E)(\alpha) = 0$ then $\alpha \in U_r(E)$ and so $\operatorname{Ker} \rho \subset f(U_r(E))$, the other inclusion being trivial. Clearly ρ is surjective and so $\bar{\rho}$ is bijective. In particular $\bar{\rho}$ induces an isomorphism

$$IJ^jT^r/IJ^jT^r \cap f(U_r(E)) = IJ^j(T^r/f(U_r(E))) \to IJ^{j+1}E$$

and it is enough to note that

$$IJ^jT^r \cap f(U_r(E)) = f(IJ^jE^r \cap U_r(E)) = f(IJ^jE^r \cap (V_r(E) + IJ^iE^r))$$

$$= f(IJ^iE^r + (IJ^jE^r \cap V_r(E))) = IJ^jT^r \cap V_r(T)$$

using Proposition 1.4(1); f commutes with the above intersection because $\operatorname{Ker} f \subset IJ^jE^r$. □

Proposition 1.6. *With the hypothesis and the notation from Lemma* 1.5, ρ *induces a surjective* Λ*-morphism* $\omega: U_r(T) \to IJ^iE$ *with* $\omega(IJ^{i-1}T^r) = IJ^iE$. *Moreover* ω *gives a surjection* $\bar{\omega}: \mathrm{Tor}_1^{\Lambda}(\bar{\Lambda}, T) \to IJ^iE$ *with* $\mathrm{Ker}\,\bar{\omega} = f(U_r(E))/V_r(T)$ *and the composite map*

$$IJ^{i-1}T^r/IJ^{i-1}T^r \cap V_r(T) \xrightarrow{v_T} \mathrm{Tor}_1^{\Lambda}(\bar{\Lambda}, T) \xrightarrow{\bar{\omega}} IJ^iE$$

is the isomorphism defined by Lemma 1.5 *for* $j = i - 1$, *the map* v_T *being induced by the inclusion* $IJ^{i-1}T^r \subset U_r(T)$.

Proof. If $f(\alpha) \in U_r(T)$, $\alpha \in E^r$, then $(\delta_1^{(r)} \otimes T)(f(\alpha)) = 0$ and so $(\delta_1^{(r)} \otimes E)(\alpha) \in IJ^iE$, i.e., $\rho(U_r(T)) \subset IJ^iE$. Thus ρ defines a map $\omega: U_r(T) \to IJ^iE$. If $y \in IJ^iE$ then $y = (\delta_1^{(r)} \otimes E)(\alpha)$ for some $\alpha \in IJ^{i-1}E^r$ and $f(\alpha) \in U_r(T)$ because $(\delta_1^{(r)} \otimes T)(f(\alpha)) = f(y) = 0$. Thus ω is surjective. Since $\mathrm{Ker}\,\rho = f(U_r(E)) \supset f(V_r(E)) = V_r(T)$, ω induces a surjection $\bar{\omega}: \mathrm{Tor}_1^{\Lambda}(\bar{\Lambda}, T) \cong U_r(T)/V_r(T) \to IJ^iE$. $\square$

Corollary 1.7. *With the hypothesis and the notation from Lemma* 1.5 *and Proposition* 1.6, *let* $S_T := IJ^{i-1}T^r/IJ^{i-1}T^r \cap V_r(T)$. Then the composite $IJ^iE \cong S_T \xrightarrow{v_T} \mathrm{Tor}_1^{\Lambda}(\bar{\Lambda}, T)$ *is a section of* $\bar{\omega}: \mathrm{Tor}_1^{\Lambda}(\bar{\Lambda}, T) \to IJ^iE$, *where the isomorphism is defined in Lemma* 1.5 *for* $j = i - 1$.

We close this section with some results concerning the infinitesimal relative★ (x, I)-liftings.

Lemma 1.8. *Let* E *be an infinitesimal relative*★ (x, I)*-lifting of* $T = E/IJ^iE$ to Λ_{i+1} *and* $\varphi \in IE[X_{s+1}, \ldots, X_r]$, $1 \le s < r$ *a homogeneous form of degree* j, $1 \le j \le i$. *Suppose that* $\varphi(x) \in (x_1, \ldots, x_s)E$. *Then there exists a homogeneous form* $\psi \in (X_1, \ldots, X_s)IE[X_1, \ldots, X_r]$ *of degree* j *such that* $\varphi(x) = \psi(x)$.

Proof. Apply induction on $t = r - s$. If $t = 1$ then $\varphi = eX_r^j$ for an $e \in IE$ and so $\varphi(x) = x_r^j e \in (x_1, \ldots, x_{r-1})E$. Since E is an infinitesimal relative ★ (x, I)-lifting of T we get $x_r^{j-1}e \in (x_1, \ldots, x_{r-1})E + IJ^iE$. Thus $x_r^{j-1}e \in x_r^i\mu + (x_1, \ldots, x_{r-1})E$ for a certain $\mu \in IE$. The homogeneous form $\eta := X_r^{j-1}(e - x_r^{i-j+1}\mu) \in IE[X_r]$ satisfies $\eta(x) \in (x_1, \ldots, x_{r-1})E$ and $\varphi(x) = x_r\eta(x)$. Thus $\eta(x) = \lambda(x)$ for a certain homogeneous form $\lambda \in E[X_1, \ldots, X_{r-1}]$ of degree 1. If $j = 1$ then $\psi := x_r\lambda \in (X_1, \ldots, X_{r-1})IE[X_1, \ldots, X_r]$ works ($J \subset I$!). Apply induction on j. Suppose $j > 1$. By induction hypothesis on j we have $\eta(x) = \theta(x)$ for a homogeneous form $\theta \in (X_1, \ldots, X_{r-1})IE[X_1, \ldots, X_r]$ of degree $j - 1$. Then $\varphi(x) = x_r\eta(x) = x_r\theta(x)$ and so $\psi := X_r\theta$ works.

Suppose now $t > 1$. Clearly φ can be written as $\varphi = \varphi' + X_{s+1}\varphi''$ where $\varphi' \in IE[X_{s+2}, \ldots, X_r]$, $\varphi'' \in IE[X_{s+1}, \ldots, X_r]$ are homogeneous forms of degree j respectively $j - 1$. Since $\varphi'(x) + x_{s+1}\varphi''(x) \in (x_1, \ldots, x_s)E$ we get $\varphi'(x) \in (x_1, \ldots, x_{s+1})E$. By induction hypothesis on t we have $\varphi'(x) = \tilde{\psi}(x)$ for a certain homogeneous form $\tilde{\psi} \in (X_1, \ldots, X_{s+1})IE[X_1, \ldots, X_r]$ of degree j. We have $\tilde{\psi} = \psi' + X_{s+1}\tilde{\psi}'$ for some homogeneous forms $\psi' \in (X_1, \ldots, X_s)IE[X_1, \ldots, X_r]$, $\tilde{\psi}' \in IE[X_{s+1}, \ldots, X_r]$ of degree j respectively $j-1$. It follows

$$x_{s+1}(\tilde{\psi}'(x) + \varphi''(x)) \in (x_1, \ldots, x_s)E$$

and so $\tilde{\psi}'(x) + \varphi''(x)) \in (x_1, \ldots, x_s)E + IJ^iE$, E being an infinitesimal relative★ (x, I)-lifting of T. Then there exists a form $\theta' \in (x_{s+1}, \ldots, x_r)^{i-j+1}IE[X_{s+1}, \ldots, X_r]$ of degree $j-1$ such that $\tilde{\psi}'(x) + \varphi''(x) - \theta'(x) \in (x_1, \ldots, x_s)E$. The homogeneous form $\eta' = \tilde{\psi}' + \varphi'' - \theta' \in IE[X_{s+1}, \ldots, X_r]$ of degree $j-1$ satisfies $\eta'(x) \in (x_1, \ldots, x_s)E$ and $\varphi(x) = \psi'(x) + x_{s+1}\eta'(x)$. Thus $\eta'(x) = \lambda'(x)$ for a certain homogeneous form $\lambda' \in E[X_1, \ldots, X_s]$ of degree 1. If $j = 1$ then $\psi = \psi' + x_{s+1}\lambda' \in (X_1, \ldots, X_s)IE[X_1, \ldots, X_r]$ works. Apply induction on j. Suppose $j > 1$. By induction hypothesis on j we have $\eta'(x) = \psi''(x)$ for a certain homogeneous form $\psi'' \in X_1, \ldots, X_s)IE[X_1, \ldots, X_r]$ of degree $j-1$. Then

$$\begin{aligned}(\psi' + X_{s+1}\psi'')(x) &= \psi'(x) + x_{s+1}\eta'(x) = \psi'(x) + x_{s+1}\tilde{\psi}'(x) + x_{s+1}\varphi''(x)\\ &= \tilde{\psi}(x) + x_{s+1}\varphi''(x) = \varphi'(x) + x_{s+1}\varphi''(x) = \varphi(x).\end{aligned}$$

As $\psi := \psi' + X_{s+1}\psi'' \in (X_1, \ldots, X_s)IE[X_1, \ldots, X_r]$, we are done. □

Proposition 1.9. *Let E be an infinitesimal relative★ (x, I)-lifting of $T = E/IJ^iE$ to A_{i+1} and j, s two integers, $1 \le s \le r$, $1 \le j < i$. Then*
(1) $V_s(E) \cap IJ^{j+1}E^s = IJ^jV_s(E)$,
(2) $U_s(E) \cap IJ^{j+1}E^s = IJ^jV_s(E) + IJ^iE^s$.

Proof. (1) Apply induction on s. If $s = 1$ there exist nothing to show. Suppose $s > 1$ and let $\alpha = (\alpha_1 | \cdots | \alpha_s) \in V_s(E) \cap IJ^{j+1}E^s$. Thus $\alpha_s \in IJ^{j+1}E$ and there exists a homogeneous form $\varphi \in IE[X_s, \ldots, X_r]$ of degree $j+1$ such that $\alpha_s - \varphi(x) \in (x_1, \ldots, x_{s-1})IJ^jE$. Since $\alpha \in V_s(E)$ there exist some $\rho_t \in E$ such that α has the following form

$$(-x_s|0 \ldots 0|x_1)\rho_1 + \cdots + (0 \ldots 0|-x_s|x_{s-1})\rho_{s-1} + (-x_{s-1}|0 \ldots 0|x_1|0)\rho_s + \cdots.$$

As the only first $(s-1)$-tuples have nonzero elements on the last position, we get $\alpha_s = \sum_{k=1}^{s-1} x_k\rho_k$ and so $\varphi(x) \in (x_1, \ldots, x_{s-1})E$. By Lemma 1.8 there exists a homogeneous form $\psi \in (X_1, \ldots, X_{s-1})IE[X_1, \ldots, X_r]$ of degree $j+1$ such that $\varphi(x) = \psi(x)$. Thus $\alpha_s, \varphi(x) \in (x_1, \ldots, x_{s-1})IJ^jE$ and we have

$$\alpha_s = \sum_{k=1}^{s-1} x_k v_k$$

for some $v_k \in IJ^jE$. Note that

$$\alpha' := \alpha - (-x_s|0 \ldots 0|x_1)v_1 - \cdots - (0 \ldots 0| - x_s|x_{s-1})v_{s-1}$$

satisfies $\alpha' - \alpha \in IJ^j V_s(E)$ and $\alpha'_s = 0$. Then $(\alpha'_1 | \ldots | \alpha'_{s-1}) \in V_{s-1}(E) \cap IJ^{j+1} E^{s-1}$ and by induction hypothesis we get $(\alpha'_1 | \ldots | \alpha'_{s-1}) \in IJ^j V_{s-1}(E)$. Hence $\alpha \in IJ^j V_s(E)$, i.e. the induction $\subset$ holds in (1), the other being trivial.

(2) By Proposition 1.4(1) we have

$$U_s(E) \cap IJ^{j+1} E^s = (V_s(E) + IJ^i E^s) \cap IJ^{j+1} E^s$$
$$= IJ^i E^s + (V_s(E) \cap IJ^{j+1} E^s).$$

Now it is enough to apply (1). □

Lemma 1.10. *Let E be an infinitesimal relative*★ *(x, I)-lifting of $T = E/IJ^i E$ to Λ_{i+1}, j, s integers, $1 \le j < i$, $1 \le s < r$ and $\varepsilon_s : E^s \to E^{s+1}$ be the map $(\alpha_1 | \ldots | \alpha_s) \to (\alpha_1 | \ldots \alpha_s | 0)$. Then $\varepsilon_s^{-1}(IJ^j V_{s+1}(E)) = IJ^j V_s(E)$.*

Proof. Let α be an element from $\varepsilon_s^{-1}(IJ^j V_{s+1}(E))$. We have

$$\varepsilon_s(\alpha) = (-x_{s+1} | 0 \ldots 0 | x_1)\rho_1 + \cdots + (0 \ldots 0 | -x_{s+1} | x_s)\rho_s$$
$$+ (-x_s | 0 \ldots 0 | x_1 | 0)\rho_{s+1} + \cdots \quad (1)$$

for some $\rho_t \in IJ^j E$. As the only first (s)-tuples have nonzero elements on the last position, we get $\sum_{t=1}^{s} x_t \rho_t = 0$. Thus $\rho = (\rho_1 | \ldots | \rho_s) \in U_s(E) \cap IJ^j E^s = IJ^{j-1} V_s(E) + IJ^i E^s$. Clearly we may change ρ adding an element from $IJ^i E^s$ because $IJ^i E^s$ is killed by multiplication with x. Then we may suppose $\rho \in IJ^{j-1} V_s(E)$. But (1) says that $\alpha + x_{s+1}\rho \in IJ^j V_s(E)$. Hence $\alpha \in IJ^j V_s(E)$. Thus the inclusion $\subset$ holds, the other being trivial. □

2. The existence of infinitesimal relative sequence

Let M be as usual a finite $\bar{\Lambda}$-module and T a Λ_i-module, $i \ge 1$ such that $T/JT \cong M$. Let $S_T := IJ^{i-1} T^r / IJ^{i-1} T^r \cap V_r(T)$ and $v_T : S_T \to \operatorname{Tor}_1^\Lambda(\bar{\Lambda}, T)$ be the injective map induced by the inclusion $IJ^{i-1} T^r \subset U_r(T)$ (see Proposition 1.6). Let $T_1 = T/IJT$ and $v_{T_1} : S_{T_1} \to \operatorname{Tor}_1^\Lambda(\bar{\Lambda}, T_1)$ be the injective map defined in Proposition 1.6. Denote $N := \operatorname{Coker} v_{T_1}$.

Lemma 2.1. *Suppose that there exists an infinitesimal relative (x, I)-lifting E of T to Λ_{i+1} and let $f : E \to T$ be the canonical surjection. Then*

(1) *the following sequence*

$$0 \to S_E \xrightarrow{v_E} \operatorname{Tor}_1^\Lambda(\bar{\Lambda}, E) \xrightarrow{\tilde{f}} \operatorname{Tor}_1^\Lambda(\bar{\Lambda}, T) \xrightarrow{\bar{\omega}} IJ^i E \to 0$$

is exact, where $\tilde{f}$ is induced by f and $\bar{\omega}$ is defined in Proposition 1.6,

(2) *$\bar{\omega}$ has a section induced by v_T,*

(3) *If $T/IJ^{j+1}T$ is an infinitesimal relative (x, I)-lifting of $T/IJ^j T$ to Λ_{j+1} for each j, $1 \le j < i$ then $\operatorname{Im} \tilde{f} \cong N$.*

Proof. Tensorizing with $\bar{A}$ the exact sequence

$$0 \to IJ^iE \xrightarrow{g} E \xrightarrow{f} T \to 0$$

we get the following exact sequence

$$\mathrm{Tor}_1^A(\bar{A}, IJ^iE) \xrightarrow{\bar{g}} \mathrm{Tor}_1^A(\bar{A}, E) \xrightarrow{\bar{f}} \mathrm{Tor}_1^A(\bar{A}, T) \xrightarrow{h} IJ^iE \to M \cong M \to 0.$$

We have $\mathrm{Tor}_1^A(\bar{A}, IJ^iE) \cong IJ^iE^r$ and so $\mathrm{Im}\,\bar{g} \cong IJ^iE^r/V_r(E) \cap IJ^iE^r = S_E$. The map h is surjective and $\mathrm{Ker}\,h = \mathrm{Im}\,\bar{f} = f(U_r(E))/V_r(T) = \mathrm{Ker}\,\bar{\omega}$ (see Proposition 1.6). Thus h and $\bar{\omega}$ coincide modulo an isomorphism of IJ^iE and the above sequence gives the exact sequence from (1). Clearly (2) follows from Corollary 1.7. Denote $N_E := \mathrm{Coker}\,v_E$, $N_T := \mathrm{Coker}\,v_T$. By (1) we have $\mathrm{Im}\,\bar{f} \cong N_E$ and the sequence

$$0 \to N_E \to \mathrm{Tor}_1^A(\bar{A}, T) \xrightarrow{\bar{\omega}} IJ^iE \cong S_T \to 0$$

is exact and split because $v_T := S_T \to \mathrm{Tor}_1^A(\bar{A}, T)$ gives a section to $\bar{\omega}$. It follows $N_E \cong \mathrm{Coker}\,v_T = N_T$. Since T is an infinitesimal relative (x, I)-lifting of $T_{i-1} := T/IJ^{i-1}T$ we get $N_T \cong N_{T_{i-1}}$. By recurrence we get $N_E \cong N_{T_1} = N$. □

Let s, $1 \le s \le r$, $S_s(T) = IJ^{i-1}T^s/IJ^{i-1}T^s \cap V_s(T)$ ($S_r(T) = S_T$ as in Corollary 1.7), $A_s := A/(x_1, \ldots, x_s)$ and $\lambda_s(T): S_s(T) \to S_T$ the map induced by the canonical inclusion $\mu_s(T): T^s \to T^r$ given by $(\alpha_1 | \ldots | \alpha_s) \to (\alpha_1 | \ldots | \alpha_s | 0 \ldots 0)$. Like v_T from Corollary 1.7 we have a natural inclusion $v_s(T): S_s(T) \to \mathrm{Tor}_1^A(A_s, T)$ and so the $S_s(T)$ are all A_s-modules. In fact the $S_s(T)$ are $\bar{A}$-modules because they are quotients of the $\bar{A}$-modules $IJ^{i-1}T^s$, $1 \le s \le r$. Let

$$0 \to S_T \xrightarrow{q} E \xrightarrow{w} T \to 0$$

be a short exact sequence of A-modules. Tensorizing with $\bar{A}$ we get the following exact sequence

$$\mathrm{Tor}_1^A(\bar{A}, E) \xrightarrow{\tilde{w}} \mathrm{Tor}_1^A(\bar{A}, T) \xrightarrow{h} S_T \xrightarrow{\bar{q}} E/JE \xrightarrow{\bar{w}} T/JT \cong M \to 0.$$

Lemma 2.2. *Suppose that T is an infinitesimal relative (x, I)-lifting of $T_{i-1} := T/IJ^{i-1}T$ to A_i if $i \ge 2$. Then the following statements are equivalent:*

(1) *E is an infinitesimal relative (x, I)-lifting of T to A_{i+1} and* $\mathrm{Im}\,q = IJ^iE$,

(2) *h is a retraction of v_T and $\lambda_s(T)$ is injective for every s, $1 \le s < r$.*

Proof. (2) ⇒ (1): By hypothesis h is surjective and so $\bar{q} = 0$, i.e. $\bar{w}$ is an isomorphism. In particular $\mathrm{Ker}\,w = \mathrm{Im}\,q \subset JE$. We will see that $\mathrm{Ker}\,w \subset IJ^iE + J\,\mathrm{Ker}\,w$, which by Nakayama's Lemma will give $\mathrm{Ker}\,w \subset IJ^iE$, i.e. $\mathrm{Ker}\,w = IJ^iE$, the other inclusion being trivial. Let $e \in \mathrm{Ker}\,w \subset JE$. Then $e = \sum_{t=1}^r x_t u_t$ for an element $u = (u_1 | \ldots | u_r) \in E^r$. Thus $w(u) \in U_r(T)$. Since h is a retraction of v_T by hypothesis we get

$$U_r(T)/V_r(T) = \mathrm{Ker}\,h + (IJ^{i-1}T^r + V_r(T))/V_r(T).$$

But $\operatorname{Ker} h = \operatorname{Im} w = w(U_r(E))/V_r(T)$ and so it follows

$$w(U_r(E)) + IJ^{i-1}T^r = U_r(T)$$

because $V_r(T) = w(V_r(E)) \subset w(U_r(E))$. Then $w(u) \in w(U_r(E)) + IJ^{i-1}T^r$ and we get $u \in IJ^{i-1}E^r + U_r(E) + (\operatorname{Ker} w)^r$. Thus $e \in IJ^iE + J \operatorname{Ker} w$.

Hence $T \cong E/IJ^iE$ and it remains to show by Proposition 1.4(1) that for every s, $1 \le s \le r$ it holds

$$U_s(E) \cap IE^s = V_s(E) + IJ^iE^s. \qquad (*)$$

As T is an infinitesimal relative (x, I)-lifting of T_{i-1} we get

$$U_s(T) \cap IT^s = V_s(T) + IJ^{i-1}T^s$$

for each s, $1 \le s \le r$ (if $i = 1$ then $(*)$ holds obviously). It follows

$$U_s(E) \cap IE^s \subset V_s(E) + IJ^{i-1}E^s$$

and the inclusion $\subset$ in $(*)$ holds if

$$(V_s(E) + IJ^{i-1}E^s) \cap U_s(E) \subset V_s(E) + IJ^iE^s.$$

But $(V_s(E) + IJ^{i-1}E^s) \cap U_s(E) = V_s(E) + (IJ^{i-1}E^s \cap U_s(E))$ because $V_s(E) \subset U_s(E)$. Thus we should show that

$$IJ^{i-1}E^s \cap U_s(E) \subset V_s(E) + IJ^iE^s$$

or equivalently

$$IJ^{i-1}T^s \cap w(U_s(E)) \subset V_s(T). \qquad (**)$$

As above $U_r(T)/V_r(T)$ is a direct sum of $IJ^{i-1}T^r + V_r(T))/V_r(T) \cong S_T$ and $\operatorname{Ker} h = w(U_r(E))/V_r(T)$. Thus we get

$$IJ^{i-1}T^r \cap w(U_r(E)) \subset V_r(T),$$

i.e. $(**)$ holds for $s = r$. It follows

$$IJ^{i-1}T^s \cap w(U_s(E)) \subset \mu_s(T)^{-1}(IJ^{i-1}T^r \cap w(U_r(E)))$$
$$\subset \mu_s(T)^{-1}(V_r(T) \cap IJ^{i-1}T^r) \subset V_s(T) \cap IJ^{i-1}T_s,$$

because $\lambda_s(T)$ is injective, in particular $(**)$ holds. Since the other inclusion in $(*)$ is trivial we are done.

(1) $\Rightarrow$ (2): As $\operatorname{Im} q = IJ^iE$, h is surjective, the morphisms $\bar{\omega}$, h coincide modulo an isomorphism defined by q and so h is a retraction of v_T by Lemma 2.1. After Proposition 1.6 let ω be the composite map $U_r(T) \to U_r(T)/V_r(T) \cong \operatorname{Tor}_1^A(\bar{A}, T) \xrightarrow{\bar{\omega}} IJ^iE$. We have $\mu_s(T)^{-1}(\operatorname{Ker} \omega) = \mu_s(T)^{-1}(w(U_r(E))) = w(U_s(E))$. Indeed if $\alpha = (\alpha_1 | \dots | \alpha_s) \in T^s$ satisfies $\mu_s(T)(\alpha) \in w(U_r(E))$ then there exists $\beta = (\beta_1 | \dots | \beta_r) \in U_r(E)$ such that $w(\beta_j)$ is α_j if $j \le s$, otherwise 0. Thus $\beta_j \in IJ^iE$ for $j > s$ and so $0 = \sum_{j=1}^r x_j\beta_j = \sum_{j=1}^s x_j\beta_j$, i.e. $\beta' = (\beta_1 | \dots | \beta_s) \in U_s(E)$. Consequently $\alpha = w(\beta') \in w(U_s(E))$.

Thus $\mathrm{Ker}(\bar{\omega}\lambda_s'(T)) = w(U_s(E))/V_s(T)$, where $\lambda_s'(T)$ is the map $\mathrm{Tor}_1^{\Lambda}(\Lambda_s, T) \to \mathrm{Tor}_1^{\Lambda}(\bar{\Lambda}, T)$ induced by $\mu_s(T)$, in fact $\lambda_s'(T) = \mathrm{Tor}_1^{\Lambda}(p_s, T)$, $p_s : \Lambda_s \to \bar{\Lambda}$ being the canonical surjection. Hence

$$\begin{aligned}\mathrm{Ker}(\bar{\omega}\nu_T\lambda_s(T)) &= \mathrm{Ker}(\bar{\omega}\lambda_s'(T)\nu_s(T)) \\ &= (w(U_s(E)) \cap IJ^{i-1}T^s)/V_s(T) \cap IJ^{i-1}T^s.\end{aligned}$$

As E is infinitesimal relative (x,I)-lifting of T we get $w(U_s(E)) \cap IJ^{i-1}T^s = w(U_s(E) \cap IE^s) \cap IJ^{i-1}T^s = w(V_s(E) + IJ^iE^s) \cap IJ^{i-1}T^s = V_s(T) \cap IJ^{i-1}T^s$. Thus $\bar{\omega}\nu_T\lambda_s(T)$ is injective and so $\lambda_s(T)$ is injective too (in fact $\bar{\omega}\nu_T\lambda_s(T)$ gives an isomorphism of $S_s(T)$ on $(x_1, \dots, x_s)IJ^{i-1}E$ and the inclusion $(x_1, \dots, x_s)\ IJ^{i-1}E \subset IJ^iE$ corresponds to $\lambda_s(T)$). □

Let

$$0 \to \Omega_\Lambda(T) \to P \to T \to 0 \qquad (+)$$

be the exact sequence defining the first *syzygy* of T over Λ (P is the free cover of T over Λ). Tensorizing (+) with $\bar{\Lambda}$ we get the following exact sequence:

$$(\xi_T) \qquad 0 = \mathrm{Tor}_1^{\Lambda}(\bar{\Lambda}, P) \to \mathrm{Tor}_1^{\Lambda}(\bar{\Lambda}, T) \xrightarrow{\alpha} \Omega_\Lambda(T)/J\Omega_\Lambda(T) \to P/JP \to M \to 0.$$

Proposition 2.3. *Suppose that T is an infinitesimal relative (x,I)-lifting of the Λ_{i-1}-module $T_{i-1} := T/IJ^{i-1}T$ to Λ_i if $i \geq 2$. Then the following statements are equivalent:*

(1) *There exists an infinitesimal relative (x,I)-lifting E of T to Λ_{i+1},*

(2) *There exists a Λ-morphism $\beta : \Omega_\Lambda(T)/J\Omega_\Lambda(T) \to S_T$ such that $\beta\alpha$ is a retraction of ν_T and $\lambda_s(T)$ is injective for all s, $1 \leq s < r$.*

Proof. (1) $\Rightarrow$ (2): Let E be an infinitesimal relative (x,I)-lifting of T to Λ_{i+1} and $f : E \to E/IJ^iE \cong T$ the canonical surjection. Since P is free we can construct the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega_\Lambda(T) & \longrightarrow & P & \longrightarrow & T & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & IJ^iE & \longrightarrow & E & \longrightarrow & T & \longrightarrow & 0 \end{array}$$

where the last vertical map is the identity. Tensorizing with $\bar{\Lambda}$ we get the following commutative diagram:

$$(\xi_T) \qquad \begin{array}{ccccccccc} 0 \longrightarrow \mathrm{Tor}_1^{\Lambda}(\bar{\Lambda}, T) & \xrightarrow{\ \alpha\ } & \Omega_\Lambda(T)/J\Omega_\Lambda(T) & \longrightarrow & P/JP & \longrightarrow & M & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \mathrm{Tor}_1^{\Lambda}(\bar{\Lambda}, T) & \xrightarrow{\ h\ } & IJ^iE & \longrightarrow & E/JE & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

with the exact rows, the first and the last vertical maps being identities. We have $S_T \cong IJ^iE$ by Lemma 1.5. Using Lemma 2.2 h induces a retraction of ν_T and $\lambda_s(T)$ is injective for every s, $1 \le s < r$. Clearly the composite map $\beta : \Omega_\Lambda(T)/J\Omega_\Lambda(T) \xrightarrow{\tau} IJ^iE \cong S_T$ works, where τ is the second vertical map in the above diagram.

(2) $\Rightarrow$ (1): Let q be the composite map $\Omega_\Lambda(T) \to \Omega_\Lambda(T)/J\Omega_\Lambda(T) \xrightarrow{\beta} S_T$, where the first map is the canonical surjection. We construct the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega_\Lambda(T) & \longrightarrow & P & \longrightarrow & T & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S_T & \xrightarrow{g} & E & \xrightarrow{f} & T & \longrightarrow & 0
\end{array}$$

where q is the first vertical map, the first square is cocartesian, the last vertical map is the identity and f is uniqually defined by the commutativity of the diagram. Clearly, the rows are exact sequences and $IJ^{i+1}E = 0$, i.e., E is in fact a finite Λ_{i+1}-module. Tensorizing by $\bar{\Lambda}$ the previous diagram we get

$$\begin{array}{lcccccccccc}
(\xi_T) & 0 \longrightarrow & \mathrm{Tor}_1^\Lambda(\bar{\Lambda}, T) & \xrightarrow{\alpha} & \Omega_\Lambda(T)/J\Omega_\Lambda(T) & \longrightarrow & P/JP & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \mathrm{Tor}_1^\Lambda(\bar{\Lambda}, T) & \xrightarrow{h} & S_T & \xrightarrow{g} & E/JE & \xrightarrow{f} & M & \longrightarrow & 0
\end{array}$$

where β is the second vertical map, the first and the last vertical ones being identities. By assumptions $(\lambda_s(T))_s$ are injective and $h = \beta\alpha$ is a retraction of ν_T. Thus E is an infinitesimal relative (x, I)-lifting by Lemma 2.2. □

Corollary 2.4. *Suppose that $T_{j+1} := T/IJ^{j+1}T$ is an infinitesimal relative (x, I)-lifting of $T_j := T/IJ^jT$ to Λ_{j+1} for each j, $1 \le j < i$, $\mathrm{Ext}^1_{\bar{\Lambda}}(N, S_T) = 0$, $\mathrm{Ext}^2_{\bar{\Lambda}}(M, S_T) = 0$ and $(\lambda_s(T))_{1 \le s < r}$ are injective. Then there exists an infinitesimal relative (x, I)-lifting of T to Λ_{i+1}.*

Proof. By Lemma 2.1 we have the following exact sequence:

$$0 \to S_T \xrightarrow{\nu_T} \mathrm{Tor}_1^\Lambda(\bar{\Lambda}, T) \to N \to 0$$

which splits because $\mathrm{Ext}^1_{\bar{\Lambda}}(N, S_T) = 0$. Let h be a retraction of ν_T and

$$(\xi_T) \qquad 0 \to \mathrm{Tor}_1^\Lambda(\bar{\Lambda}, T) \xrightarrow{\alpha} \Omega_\Lambda(T)/J\Omega_\Lambda(T) \to P/JP \to M \to 0$$

the exact sequence associated to T as above. Since P was a projective cover of T over Λ we get also that P/JP is a projective cover of M over $\bar{\Lambda}$. Thus (ξ_T) defines a short exact sequence

$$(\xi'_T) \qquad 0 \to \mathrm{Tor}_1^\Lambda(\bar{\Lambda}, T) \xrightarrow{\alpha} \Omega_\Lambda(T)/J\Omega_\Lambda(T) \to \Omega_{\bar{\Lambda}}(M) \to 0,$$

where $\Omega_{\bar\Lambda}(M)$ is the first syzygy of M over $\bar\Lambda$. Since $\mathrm{Ext}^1_{\bar\Lambda}(\Omega_{\bar\Lambda}(M), S_T) \cong \mathrm{Ext}^2_{\bar\Lambda}(M, S_T) = 0$ we get $\mathrm{Ext}^1_{\bar\Lambda}(\Omega_{\bar\Lambda}(M), h)(\xi'_T) = 0$ and so we have the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Tor}_1^{\Lambda}(\bar\Lambda, T) & \xrightarrow{\alpha} & \Omega_\Lambda(T)/J\Omega_\Lambda(T) & \longrightarrow & \Omega_{\bar\Lambda}(M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S_T & \xrightarrow{\alpha'} & C & \longrightarrow & \Omega_{\bar\Lambda}(M) & \longrightarrow & 0
\end{array}$$

where h is the first vertical map, the first square is cocartesian and α' has a retraction p. Let h' be the second vertical map above. Then $\beta = ph'$ satisfies the condition (2) from Proposition 2.3. Indeed, we have $\beta\alpha = p(h'\alpha) = p\alpha' h = h$, h being a retraction of ν_T. Applying Proposition 2.3 we are done. □

Remark 2.5. The above corollary gives conditions on T for having an infinitesimal relative (x, I)-lifting to Λ_{i+1}. Unfortunately we are not able to write on T conditions for having infinitesimal relative (x, I)-liftings to Λ_{i+1} which have still infinitesimal relative (x, I)-liftings to Λ_{i+2}. Next we will see that it is possible in the frame of infinitesimal relative★ (x, I)-liftings.

Lemma 2.6. *Suppose that $T_{j+1} := T/IJ^{j+1}T$ is an infinitesimal relative★ (x, I)-lifting of $T_j = T/IJ^jT$ to Λ_{j+1} for each j, $1 \le j < i$, $\mathrm{Ext}^1_{\bar\Lambda}(N, S_T) = 0$, $\mathrm{Ext}^2_{\bar\Lambda}(M, S_T) = 0$. Then there exists an infinitesimal relative (x, I)-lifting of T to Λ_{i+1}.*

Proof. By Corollary 2.4 it is enough to show that $(\lambda_s(T))_{1 \le s < r}$ are injective. Using Proposition 1.9(1) it is enough to show that

$$\mu_s(T)^{-1}(IJ^{i-2}V_r(T)) = IJ^{i-2}V_s(T).$$

But this follows applying successively Lemma 1.10. □

Let T_2 be an arbitrary finite Λ_2-module such that $T_2/IJT_2 \cong M$ and N the cokernel of $\nu_{T_2}: S_{T_2} \to \mathrm{Tor}_1^{\Lambda}(\bar\Lambda, T_2)$. Let $\mathcal{T}_\Lambda(J) := \oplus_{n \ge 0}(\otimes^n J)$ be the tensor algebra of J, $K := \mathcal{T}_\Lambda(J) \otimes_\Lambda T_2$, $K = \oplus_{n \ge 1} K_n$, $K_{n+1} := (\otimes^n J) \otimes_\Lambda T_2$ and $S := \oplus_{n \ge 2} S_n$, $S_n := IJK_n$. Note that S is a graded Λ-module ($IJ^2K = 0$!) and $K_{n+1} = J \otimes K_n \cong K_n^r / V_r(K_n)$, the isomorphism follows tensorizing by K_n the exact sequence

$$\Lambda^{\binom{r}{2}} \xrightarrow{\delta_2^{(r)}} \Lambda^r \to J \to 0.$$

Theorem 2.7. *Suppose that T_2 is an infinitesimal relative★ (x, I)-lifting of M to Λ_2 and* $\mathrm{Ext}^1_{\bar\Lambda}(N, S) = 0$, $\mathrm{Ext}^2_{\bar\Lambda}(M, S) = 0$. *Then there exists a sequence of finite Λ-modules $(T_i)_{i \ge 3}$ such that for all $i \ge 3$*

(1) *T_i is a Λ_i-module,*
(2) *T_i is an infinitesimal relative★ (x, I)-lifting of T_{i-1} to Λ_i,*
(3) *$S_{i-1} \cong IJ^{i-1}T_i$.*

Proof. Apply induction on $i \geq 2$. Suppose that T_j, $2 \leq j < i$ are already found. We have

$$S_{T_{i-1}} = IJ^{i-2}T^r_{i-1}/IJ^{i-2}T^r_{i-1} \cap V_r(T_{i-1})$$
$$= IJ^{i-2}(T^r_{i-1}/V_r(T_{i-1})) = IJ^{i-2}(J \otimes_A T_{i-1}).$$

If $i = 3$ it follows $S_{T_2} = IJK_2 = S_2$. If $i > 3$ then we see that

$$IJ^2(J \otimes_A T_{i-1}) \cong IJ^2T^r_{i-1}/IJ^2T^r_{i-1} \cap V_r(T_{i-1})$$
$$= IJ^2T^r_{i-1}/IJV_r(T_{i-1}) \cong IJ(J \otimes_A (JT_{i-1}))$$

(see Proposition 1.9(1)) though $IJ(J \otimes_A T_{i-1})$ may be not isomorphic with $I(J \otimes_A (JT_{i-1}))$ if $I \neq J$. Thus

$$S_{T_{i-1}} = IJ^{i-2}(J \otimes_A T_{i-1}) \cong IJ^{i-3}(J \otimes_A (JT_{i-1})) \cong IJ^{i-3}(J \otimes_A (J \otimes_A T_{i-2}))$$
$$\cong \cdots \cong IJ((\otimes^{i-2}J) \otimes_A T_2) = IJK_{i-1} = S_{i-1}$$

because $IJ^2T_{i-1} \cong IJ(J \otimes_A T_{i-2})$ by Lemma 1.5. By assumption we have $\mathrm{Ext}^1_{\bar{A}}(N, S_{i-1}) = 0$, $\mathrm{Ext}^2_{\bar{A}}(M, S_{i-1}) = 0$ and so there exists an infinitesimal relative★ (x, I)-lifting T_i of T_{i-1} to A_i (see Lemma 2.6). We have $IJ^{i-1}T_i \cong S_{T_{i-1}}$ by Lemma 1.5 and thus T_i satisfies 3). Remains to show that $\bar{T}_i := T_i/(x_1, \ldots, x_s)T_i$ is an infinitesimal relative (x, I)-lifting of $\bar{T}_{i-1} := T_{i-1}/(x_1, \ldots, x_s)T_{i-1}$, $1 \leq s < r$ (see Proposition 1.4(2)).

We have the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S_{i-1} & \longrightarrow & T_i & \xrightarrow{f} & T_{i-1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & S_{\bar{T}_{i-1}} & \longrightarrow & \bar{T}_i & \xrightarrow{\bar{f}} & \bar{T}_{i-1} & \longrightarrow & 0 \end{array} \qquad (*)$$

where the rows are exact, the last two vertical maps are canonical surjections and induce the first vertical map τ. The map τ is surjective because of the Snake Lemma, f inducing a surjective map $(x_1, \ldots, x_s)T_i \to (x_1, \ldots, x_s)T_{i-1}$. Tensorizing by $\bar{A}$ over A we get the following commutative diagram

$$\begin{array}{ccccccccc} \mathrm{Tor}_1^A(\bar{A}, T_i) & \xrightarrow{\tilde{f}} & \mathrm{Tor}_1^A(\bar{A}, T_{i-1}) & \xrightarrow{h} & S_{i-1} & \longrightarrow & M \cong M & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \quad \downarrow & & \\ \mathrm{Tor}_1^A(\bar{A}, \bar{T}_i) & \xrightarrow{\hat{f}} & \mathrm{Tor}_1^A(\bar{A}, \bar{T}_{i-1}) & \xrightarrow{\hat{h}} & S_{\bar{T}_{i-1}} & \longrightarrow & M \cong M & \longrightarrow & 0 \end{array} \qquad (**)$$

where the rows are exact. It follows $h, \hat{h}$ are surjective and h is a retraction of $v_{T_{i-1}}$ because T_i is an infinitesimal relative (x, I)-lifting of T_{i-1} (see Lemma 2.2).

Let $A_s := A/(x_1, \ldots, x_s)$. Tensorizing the bottom exact sequence from $(*)$ by $\bar{A}$ over A_s we get the following exact sequence

$$\mathrm{Tor}_1^{A_s}(\bar{A}, \bar{T}_i) \xrightarrow{f'} \mathrm{Tor}_1^{A_s}(\bar{A}, \bar{T}_{i-1}) \xrightarrow{h'} S_{\bar{T}_{i-1}} \to M \cong M \to 0$$

By Lemma 2.2 it is enough to show that $\lambda_e(\bar{T}_{i-1})$ are injective for $1 \leq e < r - s$ and h' is a retraction of $v_{\bar{T}_{i-1}}: S_{\bar{T}_{i-1}} \to \mathrm{Tor}_1^{\Lambda_s}(\bar{\Lambda}, \bar{T}_{i-1})$, the first condition being a consequence of Lemma 1.10, $\bar{T}_{i-1}$ being an infinitesimal relative★ (x, I)-lifting of $\bar{T}_{i-2}$ by induction hypothesis.

Remains to show that $h' v_{\bar{T}_{i-1}} = 1$. For this purpose we need a carefull examination of $(**)$. Note that $\mathrm{Tor}_1^{\Lambda_s}(\bar{\Lambda}, \bar{T}_i) = \bar{U}/\bar{V}$, where $\bar{U}$ is the set of all tuples $(\bar{u}_{s+1} | \dots | \bar{u}_r) \in \bar{T}_i^{r-s}$ such that $\sum_{t=s+1}^r x_t \bar{u}_t = 0$ and

$$\bar{V} = \langle (-x_r | 0 \dots 0 | x_{s+1}), \dots, (0 \dots 0 | -x_r | x_{r-1}),$$

$$(-x_{r-1} | 0 \dots 0 | x_{s+1} | 0), \dots \rangle \bar{T}_i.$$

Since $x_1, \dots, x_s$ act trivially on $\bar{T}_i$ we have $U_r(\bar{T}_i) = (\bar{T}_i^s | \bar{U})$, $V_r(\bar{T}_i) = ((x_{s+1}, \dots, x_r)\bar{T}_i^s | \bar{V})$ (note for example that $V_r(\bar{T}_i)$ contains the submodule $(x_r | 0 \dots 0 | x_1)\bar{T}_i = (x_r \bar{T}_i | 0 \dots 0)$). Thus

$$\mathrm{Tor}_1^{\Lambda}(\bar{\Lambda}, \bar{T}_i) \cong U_r(\bar{T}_i)/V_r(\bar{T}_i) \cong (\bar{T}_i^s/(x_{s+1+}, \dots, x_r)\bar{T}_i^s \oplus \bar{U}/\bar{V}$$

$$\cong M^s \oplus \mathrm{Tor}_1^{\Lambda_s}(\bar{\Lambda}, \bar{T}_i)$$

and it is easy to see that $\hat{f} = 1_{M^s} \oplus f'$ modulo this isomorphism. It follows that $\hat{h} = h'\pi$, where $\pi: \mathrm{Tor}_1^{\Lambda}(\bar{\Lambda}, \bar{T}_{i-1}) \to \mathrm{Tor}_1^{\Lambda_s}(\bar{\Lambda}, \bar{T}_{i-1})$ is the canonical projection ($\mathrm{Ker}\, \hat{h} = \mathrm{Im}\, \hat{f}$!). Let $\hat{v}$ be the composite map $S_{\bar{T}_{i-1}} \xrightarrow{v_{\bar{T}_{i-1}}} \mathrm{Tor}_1^{\Lambda_s}(\bar{\Lambda}, \bar{T}_{i-1}) \to \mathrm{Tor}_1^{\Lambda}(\bar{\Lambda}, \bar{T}_{i-1})$, where the last map is the canonical injection $(\bar{u}_{s+1} | \dots | \bar{u}_r) \to (0 \dots 0 | \bar{u}_{s+1} | \dots | \bar{u}_r)$. Let $u = (u_1 | \dots | u_r) \in IJ^{i-2} T_{i-1}^r$ and ϕ the second vertical map from $(**)$. We have

$$(\phi v_{T_{i-1}})(\mathrm{cls.}\ u \bmod IJ^{i-3} V_r(T_{i-1})) = \mathrm{cls.}\ (0 \dots 0 | \bar{u}_{s+1} | \dots | \bar{u}_r) \bmod U_r(\bar{T}_{i-1})$$

because $V_r(\bar{T}_{i-1}) = ((x_{s+1}, \dots, x_r)\bar{T}_{i-1}^s | \bar{V})$ and $u_1, \dots, u_r \in JT_{i-1}$. On the other hand

$$\hat{v}\tau(\mathrm{cls.}\ u \bmod IJ^{i-3} V_r(T_{i-1})) = \hat{v}(\mathrm{cls.}\ (\bar{u}_{s+1} | \dots | \bar{u}_r) \bmod IJ^{i-3} \bar{V})$$

$$= \mathrm{cls.}\ (0 \dots 0 | \bar{u}_{s+1} | \dots | \bar{u}_r) \bmod U_r(\bar{T}_{i-1}).$$

Thus $\hat{v}\tau = \phi v_{T_{i-1}}$ and it follows $\hat{h}(\hat{v}\tau) = \hat{h}\phi v_{T_{i-1}} = (\tau h) v_{T_{i-1}} = \tau$. Hence $h' v_{\bar{T}_{i-1}} = \hat{h}\hat{v} = 1_{S_{\bar{T}_{i-1}}}$, τ being surjective and $\hat{h} = h'\pi$. □

Remark 2.8. If $T_2 = M$ then $JT_2 = 0$ and by our construction $S_i = 0$ for all $i \geq 1$. Then the conditions of Theorem 2.7 are trivially fulfilled if $I = J$. Indeed, the sequence $T_i := M$, $i \geq 2$ works. If $I \neq J$ then T_2 can be not an infinitesimal relative★ (x, I)-lifting of M.

Theorem 2.9. *Let T_1 be a finite Λ_1-module such that $T_1/JT_1 \cong M$, N the kernel of the surjection $\bar{\omega}_1: \mathrm{Tor}_1^{\Lambda}(\bar{\Lambda}, M) \to JT_1$ (see Proposition 1.6), $K_n' = (\otimes^n J) \otimes_\Lambda T_1$, $S_n' = JK_n'$ and $S' = \otimes_{n \geq 1} S_n'$. Suppose that $I = J$, $\mathrm{Ext}_1^1(N, S') = 0$ and $\mathrm{Ext}_{\bar{\Lambda}}^2(M, S') = 0$. Then* *there exists a sequence of finite Λ-modules* $(T_i)_{i \geq 2}$ *such that for all* $i \geq 2$;

(1) T_i *is a* Λ_i*-module,*

(2) T_i *is an infinitesimal relative*★ (x, J)*-lifting of* T_{i-1} *to* Λ_i,

(3) $S'_{i-1} \cong J^i T_i$.

The proof goes as in Theorem 2.7 but since $I = J$ we may start the induction with $i = 1$. By Remark 1.1 T_1 is already an infinitesimal relative★ (x, J)-lifting of M to Λ_1. Note also that $N \cong \operatorname{Coker} \nu_{T_1}$ by Lemma 2.1(1).

Next we try to understand which are in fact the modules $(S_{T_j})_{1 \leq j \leq i}$ if T_i is an infinitesimal lifting of M ($I = \Lambda$!), i.e. T_j is an infinitesimal lifting of $T_{j-1} := T_j/J^{j-1}T_j$ for all $2 \leq j \leq i$ ($\Lambda_1 = \bar{\Lambda}$, $T_1 = M$). Clearly T_i is an infinitesimal lifting of M if the canonical surjection $\phi: T_i[X_1, \ldots, X_r] \to \bigoplus_{t=0}^{i} J^{t-1} T_t$ given by $X \to x$ defines a graded isomorphism $\bar{\phi}: M[X_1, \ldots, X_r]/(X)^i \to \bigoplus_{t=0}^{i} J^{t-1} T_t$. In particular in this case $N = 0$.

Lemma 2.10. *Let* $s \geq 1$ *and* T_i *be an infinitesimal relative*★ (x, I)*-lifting of* $T_{i-1} := T_i/IJ^{i-1}T_i$. *Then there exists an exact sequence*

$$0 \to S_s(T_i) \xrightarrow{\gamma_s} S_{s+1}(T_i) \xrightarrow{\tau_s} IJ^{i-1}T_i/(x_1, \ldots, x_s)IJ^{i-2}T_i \to 0.$$

Proof. Let γ_s be the map induced by ε_s (see Lemma 1.10 for notation). Then γ_s is injective by Lemma 1.10. Let $p_{s+1}: T_i^{s+1} \to T_i$ be the $(s+1)$th projection. Then $S_{s+1}(T_i)/\operatorname{Im}\gamma_s \cong IJ^{i-1}T_i/p_{s+1}(IJ^{i-2}V_{s+1}(T_i))$ and remains to show that $p_{s+1}(IJ^{i-2}V_{s+1}(T_i)) = (x_1, \ldots, x_s)IJ^{i-2}T_i$. But this is obvious because the generators of $V_{s+1}(T_i)$ which are not in $\operatorname{Ker} p_{s+1}$ have the form $(0 \ldots 0| -x_{s+1}|0 \ldots 0|x_j)$ with $1 \leq j \leq s$. Thus the composite canonical map $\tau_s: S_{s+1}(T_i) \to S_{s+1}(T_i)/\operatorname{Im}\gamma_s \cong IJ^{i-1}T_i/(x_1, \ldots, x_s)IJ^{i-2}T_i$ works. □

Lemma 2.11. *Let* T_i *be an infinitesimal lifting of* M. *Then* $S_{T_i} \cong M^{\binom{r+i-1}{i}}$.

Proof. Apply induction on i. If $i = 1$ then $S_{T_1} = JT_2 \cong M^r$ because $\bar{\phi}$ above is an isomorphism. Suppose that $i > 1$. By Lemma 2.10 we have the following exact sequence

$$0 \to S_s(T_i) \xrightarrow{\gamma_s} S_{s+1}(T_i) \xrightarrow{\tau_s} J^{i-1}T_i/(x_1, \ldots, x_s)J^{i-2}T_i \to 0$$

for all s, $1 \leq s < r$. As T_i is an infinitesimal lifting of M ($\bar{\phi}$ is an isomorphism !) we have

$$J^{i-1}T_i/(x_1, \ldots, x_s)J^{i-2}T_i \cong (x_{s+1}, \ldots, x_r)^{i-1}T_i \cong M^{\binom{r-s+i-2}{i-1}}.$$

Moreover the inclusion $\theta: (x_{s+1}, \ldots, x_r)^{i-1}T_i \to J^{i-1}T_i$ defines a section for τ_s given by $u \to (0 \ldots 0|\theta(u))$ and so the above exact sequence splits. Thus

$$S_{T_i} = S_r(T_i) = S_1(T_i) \oplus \bigoplus_{s=1}^{r-1} M^{\binom{r-s+i-2}{i-1}}.$$

But $S_1(T_i) = J^{i-1}T_i \cong S_{T_{i-1}} \cong M^{\binom{r+i-2}{i-1}}$ by induction hypothesis. As $\sum_{s=0}^{r-1}\binom{r-s+i-2}{i-1} = \binom{r+i-1}{i}$ we are done. □

Proposition 2.12 (Auslander et al. [1, (1.5), (1.6)]). *Let T_2 be an infinitesimal lifting of M to Λ_2. Suppose that $\mathrm{Ext}^2_{\Lambda_1}(M, M) = 0$. Then there exists a sequence of finite Λ-modules $(T_i)_{i\geq 3}$ such that for all $i \geq 3$, T_i is an infinitesimal lifting of T_{i-1} to Λ_i.*

Proof. Apply induction on $i \geq 2$. If T_i is given then S_i is a direct sum of copies of M (see Lemma 2.11). In particular $\mathrm{Ext}^2_{\Lambda_1}(M, S_i) = 0$. Note also that the kernel N of the map $\bar{\omega}_1 : \mathrm{Tor}_1^A(\Lambda_1, M) \to JT_2$ is zero. Applying Theorem 2.7 we get an infinitesimal relative★ (x, Λ)-lifting T_{i+1} of T_i to Λ_{i+1}. The morphism $\nu_{T_{i+1}}$ from the following exact sequence given by Lemma 2.1

$$0 \to S_{T_{i+1}} \xrightarrow{\nu_{T_{i+1}}} \mathrm{Tor}_1^A(\Lambda_1, T_{i+1}) \to \mathrm{Tor}_1^A(\Lambda_1, T_i) \to S_{T_i} \to 0$$

is surjective, because $N = 0$. Thus $U_r(T_{i+1}) = J^i T^r_{i+1} + V_r(T_{i+1})$ and so T_{i+1} is an infinitesimal lifting of T_i to Λ_{i+1} (this follows as (2)⇒(1) in Lemma 1.3). □

3. Relative★ (x, I)-liftings over complete rings

Let T_2 be an infinitesimal relative★ (x, I)-lifting of M to Λ_2.

Lemma 3.1. *Suppose that A is complete in the IJ-adic topology and there exists a sequence of finite Λ-modulus $(T_i)_{i\geq 3}$ such that for each $i \geq 3$ (T_i) is an infinitesimal relative★ (x, I)-lifting of (T_{i-1}) to Λ_i. Then there exists a relative★ (x, I)-lifting T of M to Λ such that $T_2 \cong T/IJ^2T$.*

Proof (after Auslander et al. [1, (1.2)]. Fix $s \in \mathbb{N}$. We have the following commutative diagram for all $i \geq 1$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & IJ^sT_{i+s+1} & \longrightarrow & T_{i+s+1} & \longrightarrow & T_s & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & IJ^sT_{i+s} & \longrightarrow & T_{i+s} & \longrightarrow & T_s & \longrightarrow & 0 \end{array}$$

where the third vertical map is the identity and the second vertical map τ is the canonical surjection. Clearly the first vertical map γ induced by τ is also surjective and so we get an exact sequence taking projective limits

$$0 \to V_s := \varprojlim_{i\geq 2} IJ^sT_{i+s} \to T := \varprojlim_{i\geq 2} T_{i+s} \to T_s \to 0.$$

It is obvious that $V_s \supset IJ^sT$.

Now we show that T is a finite Λ-module. As M is finite there exists a surjective map $\bar{\phi}: \Lambda^t \to M$ for a certain $t \in \mathbb{N}$. Let $\phi: \Lambda^t \to T$ be a lifting of $\bar{\phi}$ to T ($T_2 \cong T/V_2$!) and $\theta_j: T \to T_j$, $j \in \mathbb{N}$ the limit maps. By Nakayama's Lemma it follows that $\phi_j = \theta_j \phi$ is surjective for all $j > s$. We have the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & IJ^i\Lambda^t/IJ^{i+1}\Lambda^t & \xrightarrow{\psi_{i+1}} & IJ^iT_{i+1} & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Ker} \bar{\phi}_{i+1} & \longrightarrow & \Lambda^t/IJ^{i+1}\Lambda^t & \xrightarrow{\bar{\phi}_{i+1}} & T_{i+1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Ker} \bar{\phi}_i & \longrightarrow & \Lambda^t/IJ^i\Lambda^t & \xrightarrow{\bar{\phi}_i} & T_i & \longrightarrow & 0
\end{array}$$

where the lines and columns are exact, $\bar{\phi}_i$ being induced by ϕ_i and the last two vertical maps from bottom are the canonical surjections. Clearly ψ_{i+1} is surjective because $\bar{\phi}_{i+1}$ is so. By the Snake Lemma we get also surjective the first vertical map from bottom. Taking projective limits we get the following exact sequence:

$$0 \longrightarrow \lim_{i \geq 2} \operatorname{Ker} \bar{\phi}_i \longrightarrow \Lambda^t \xrightarrow{\phi} T \longrightarrow 0.$$

In particular T is finite over Λ. Then IJ^sT is complete in the IJ-adic topology and so $V_s = IJ^sT$, thus $T/IJ^sT \cong T_s$. In particular $T/IJ^2T \cong T_2$.

It remains to show that

$$((x_1, \ldots, x_s)T : x_{s+1})_T \cap IT = (x_1, \ldots, x_s)T \qquad (*)$$

for all s, $0 \leq s < r$. By assumptions we have

$$((x_1, \ldots, x_s)T_{i+1} : x_{s+1})_{T_{i+1}} \cap IT_{i+1} = (x_1, \ldots, x_s)T_{i+1} + IJ^iT_{i+1}$$

for all $i \geq 1$. Thus

$$((x_1, \ldots, x_s)T : x_{s+1})_T \cap IT = (x_1, \ldots, x_s)T + IJ^iT$$

for all $i \geq 1$. As $(x_1, \ldots, x_s)T$ is closed in the IJ-adic topology we get the inclusion $\subset$ in $(*)$, the other one being trivial. □

Theorem 3.2. *Suppose that Λ is complete in the IJ-adic topology and $Ext^1_\Lambda(N, S) = 0$, $Ext^2_\Lambda(M, S) = 0$ in the notation of Theorem 2.7. Then there exists a relative*★ *(x, I)-lifting T of M to Λ such that $T/IJ^2T \cong T_2$.*

The proof follows from Theorem 2.7 with the help of Lemma 3.1.

Theorem 3.3. *With the notation and hypothesis of Theorem 3.2 suppose that x is a system of parameters in Λ and $J \subset I^2$. Then there exists a generalized maximal Cohen–Macaulay module T such that:*
(i) $I H^i_m(T) = 0$, $i \neq \dim \Lambda$,
(ii) $T/IJ^2T \cong T_2$, *in particular* $T/JT \cong M$.

Proof. By Theorem 3.2 there exists a relative* (x, I)-lifting T of T_2 to Λ. We have

$$((x_1, \dots, x_s)T : x_{s+1})_T \cap IT = (x_1, \dots, x_s)T$$

for all s, $0 \leq s < r$. Thus

$$I((x_1, \dots, x_s)T : x_{s+1})_T \subset (x_1, \dots, x_s)T$$

for all s, $0 \leq s < r$. Hence x is an I-weak T-sequence in the terminology of [9] Appendix. Then (i) follows by [9] Appendix, Lemma 12 since $J \subset I^2$. □

Corollary 3.4. *With the notation and hypothesis of Theorem 3.2 suppose that $I = m$, x is a system of parameters in Λ and $(x) \subset m^2$. Then there exists a maximal quasi-Buchsbaum Λ-module T such that $T/(x)T \cong M$.*

Theorem 3.5. *Suppose that Λ is complete in the I-adic topology, $I = J$, $\mathrm{Ext}^1_{\bar\Lambda}(N, S') = 0$ and $\mathrm{Ext}^2_{\bar\Lambda}(M, S') = 0$ in the notation and hypothesis of Theorem 2.9. Then there exists a relative* (x, J)-lifting T of M to Λ such that $T/J^2T \cong T_1$.*

The proof follows from Theorem 2.9 with the help of Lemma 3.1.
If $r = 1$ the Theorem 3.5 has the following easier form:

Theorem 3.6. *Let (Λ, m) be a complete Noetherian local ring, $x \in \Lambda$ a regular element, $I = (x)$, T_1 a finite $\Lambda_1 := \Lambda/x^2\Lambda$-module, $N = ((0) : x)_{T_1}/xT_1$, $\bar\Lambda := \Lambda/x\Lambda$ and $M = T_1/xT_1$. Suppose that $\mathrm{Ext}^1_{\bar\Lambda}(N, xT_1) = 0$, $\mathrm{Ext}^2_{\bar\Lambda}(M, xT_1) = 0$. Then there exists a relative* (x, I)-lifting T of M to Λ such that $T/x^2T \cong T_1$.*

Proof. In the notation of Theorem 2.9 we see that $K_{i+1} = (x) \otimes K_i \cong K_i$ and so $S'_{i+1} = (x)K_{i+1} \cong S'_i$ for al $i \geq 1$. Hence $S'_i \cong S'_1 = xT_1$ for all $i \geq 1$ and thus the hypothesis of Theorem 3.5 hold. □

Proposition 3.7 (Auslander et al. [1, (1.6)]). *Suppose that Λ is complete in the (x)-adic topology, $x = (x_1, \dots, x_r)$, T_2 is an infinitesimal lifting of M to Λ_2 and $\mathrm{Ext}^2_{\Lambda_1}(M, M) = 0$. Then T_2 is liftable to Λ.*

The proof follows from Proposition 2.12 and Lemma 3.1.

Example 3.8. Let k be a field, $\Lambda = k[[Y, Z]]$, $x = (x_1, x_2)$, $x_1 = Y^2$, $x_2 = Z^2$, $I = (x)$ and $m = (Y, Z)$. Then $M := m/xm$ as a $\bar\Lambda := \Lambda/(x)$-module is not liftable to Λ because

M is not free over $\bar{A}$. Indeed a lifting L of M to A must be a MCM A-module and so L is free over A. Then M is free over $\bar{A}$ which is not possible because $\dim_k \bar{A} = 4$ and $\dim_k M = 5$. Contradiction ! However m is a relative★ (x, I)-lifting of M to A because $(x_1 m \colon x_2)_m \cap xm = x_1 m$.

Example 3.9. Let A be a DVR, $x \in A$ a local parameter, $T_1 = A[[Y]]/(x^2, Y^2, xY)$, $I = (x)$ and $M = T_1/xT_1 \cong k[[Y]]/(Y^2)$, k being the residue field of A. Then $((0)\colon x)_{T_1} = (Y, x)T_1$ and $N = (Y, x)T_1/xT_1 \cong YM \cong k$. Since $\bar{A} = k$ we have $\mathrm{Ext}^1_{\bar{A}}(N, xT_2) = 0$, $\mathrm{Ext}^2_{\bar{A}}(M, xT_2) = 0$ and so T_1 is relative★ (x, I)-liftable to A. Such a relative★ (x, I)-lifting of T_1 is $T = A[[Y]]/(xY, Y^2)$. Certainly $A[[Y]]/(Y^2)$ is a lifting of M to A but there exist no lifting L of M to A such that $L/x^2L \cong T_1(((0)\colon x)_{T_1} \neq xT_1!)$.

Example 3.10. Let k be a field, $A = k[[Y, Z]]$, $x = Y^2$, $T_1 = A_1[[U]]/(ZU - x, xU, U^4)$ and $I = (x)$. Clearly T_1 is an infinitesimal relative★ (x, I)-lifting of $M = T_1/xT_1$ to A_1. However there exist no infinitesimal relative (x, I)-liftings of T_1 to A_2 because the canonical map $v_{T_1}\colon xT_1 \to ((0)\colon x)_{T_1}$ has no retractions (see Proposition 2.3). Indeed, let h be a retraction. We have $xT_1 = xA_1$ and $((0)\colon x)_{T_1} = (x, U)T_1$ because $xU = 0$ in T_1. It follows $Zh(u) = h(x) = x$ (h is a retraction of v_{T_1}!). Since $h(u) = xt$ for a certain $t \in A_1$ we get $x(Zt - 1) = 0$ in A_1 and so Z is invertible in A. Contradiction ! Thus there exist no relative (x, I)-liftings T of M to A such that $T/x^2T \cong T_1$. In particular there exist no relative★ (x, I)-liftings of T_1 to A.

Acknowledgements

The author would like to thank M. Fiorentini and A. Lascu who inspired the idea to extend the liftability from [1] in the frame of relative sequences and to J. Herzog for some very useful conversations related to Proposition 2.3, Corollary 2.4 and Theorem 2.7. This paper started while the author was visiting the University of Ferrara and was mainly written when the author was visiting the University of Essen by a grant of the Alexander von Humboldt Foundation. The last period of preparation of this paper was supported by the Romanian Institute of Mathematics, the Bucharest University and the Sonderforschungsbereich 170 in Göttingen. The author is grateful to all these institution for supports and hospitality.

References

[1] M. Auslander, S. Ding and Ø. Solberg, Liftings and weak liftings of modules, J. Algebra 156 (1993) 273–317.

[2] M. Cipu, J. Herzog and D. Popescu, Indecomposable generalized Cohen–Macaulay modules, Trans. Amer. Math. Soc. 342 (1994) 107–136.

[3] M. Fiorentini, On relative regular sequences, J. Algebra 18 (1971) 384–389.

[4] J. Herzog, A. Simis and W.V. Vasconcelos, Approximation complexes of blowing up rings I, J. Algebra 74 (1982) 466–493.
[5] C. Huneke, The theory of d-sequences and powers of ideals, Adv. Math. 46 (1982) 249–279.
[6] D. Popescu, Indecomposable Cohen–Macaulay modules and their multiplicities, Trans. Amer Math. Soc. 323 (1991) 369–387.
[7] D. Popescu, Liftings of finite injective dimension modules, Analele Univ. Bucuresti, XL (1991) 65–75.
[8] D. Popescu and M. Roczen, Indecomposable Cohen–Macaulay modules and irreducible maps, Compositio Math. 76 (1990) 277–294.
[9] J. Stückrad and W. Vogel, Buchsbaum Rings and Applications (Springer, Berlin, 1986).
[10] Y. Yoshino, Cohen–Macaulay modules over Cohen–Macaulay rings, London Math. Soc. Lecture Note Ser. Vol. 146 (Cambridge Univ. Press, Cambridge, 1990).